Physics 606 Exam 2

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work.

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

Do all your work on the blank sheets provided, writing your name clearly, and turn them in stapled together. You may keep these questions.

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\hbar\omega}}\,p \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,x - \frac{i}{\sqrt{2m\hbar\omega}}\,p \quad , \quad N = a^\dagger a$$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad , \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad \vec{p} \to -i\hbar\vec{\nabla}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad [\sigma_i, \sigma_j] = 2\,i\,\varepsilon_{ijk}\,\sigma_k$$

$$L_z\,|\ell, m\rangle = m\hbar\,|\ell, m\rangle \quad , \quad L^2\,|\ell, m\rangle = \ell\,(\ell+1)\,\hbar^2\,|\ell, m\rangle$$

$$L_\pm \equiv L_x \pm iL_y \quad , \quad L_\pm\,|\ell, m\rangle = \hbar\sqrt{\ell\,(\ell+1) - m\,(m\pm 1)}\,|\ell, m\pm 1\rangle$$

$$\left[\hat{p}_i, F(\hat{\mathbf{r}})\right] = -i\hbar\frac{\partial F}{\partial \hat{x}_i} \quad \text{and} \quad \left[\hat{x}_i, G(\hat{\mathbf{p}})\right] = i\hbar\frac{\partial G}{\partial \hat{p}_i} \quad .$$

Possibly useful:

$$\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u\, e^{-au^2+bu+c} = \frac{d}{db}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{b}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u^2\, e^{-au^2+bu+c} = \frac{d^2}{db^2}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c} + \left(\frac{b}{2a}\right)^2\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$$

1. We obtained the equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}+V(r)-E\right]u_\ell(r)=0$$

for the radial wavefunction $R_\ell(r)$ of a particle in a spherically symmetric potential $V(r)$, where

$$u_\ell(r)=rR_\ell(r)\ .$$

As usual, we assume that $V(r)$ does not diverge or diverges more slowly than $1/r^2$.

(a) (15) For $\ell\geq 1$, show that

$$u_\ell(r)\propto r^{\alpha+1}\quad\text{or}\quad R_\ell(r)\propto r^\alpha\quad\text{as}\quad r\to 0$$

where you will determine α in terms of ℓ.

(b) (10) For $\ell=0$, show that

$$R_\ell(r)\to\text{constant}$$

satisfies the equation as $r\to 0$ if $V(r)$ does not diverge or diverges more slowly than $1/r$ as $r\to 0$.

2. An electron in a hydrogen atom is in the state

$$\frac{1}{6}\left[4\,|1,0,0\rangle + 3\,|2,1,1\rangle - |2,1,0\rangle + \sqrt{10}\,|2,1,-1\rangle\right]$$

with the usual notation $|n, \ell, m\rangle$.

Calculate the following:

(a) (9) The energy in terms of the magnitude E_0 of the ground state energy.

($-E_0$ is the lowest energy, for $n = 1$.)

(b) (9) The expectation value of L^2, in terms of $\hbar$.

(Here $\vec{L}$ is the orbital angular momentum, of course.)

(c) (7) The expectation value of L_z, in terms of $\hbar$.

3. (a) (9) Explicitly calculate the 3×3 matrices that represent L_+ and L_- in the space of $\ell = 1$ functions:

$$\begin{aligned} \left(L_+\right)_{m,m'} &= \langle \ell = 1, m|\, L_+ \,|\ell = 1, m'\rangle \\ \left(L_-\right)_{m,m'} &= \langle \ell = 1, m|\, L_- \,|\ell = 1, m'\rangle \ . \end{aligned}$$

At the end, write both of these in the usual form of 3×3 matrices.

(b) (9) Using your results from part (a), explicitly calculate the 3×3 matrices that represent L_x, L_y, and L_z, again in the space of $\ell = 1$ functions:

$$\begin{aligned} \left(L_x\right)_{m,m'} &= \langle \ell = 1, m|\, L_x \,|\ell = 1, m'\rangle \\ \left(L_y\right)_{m,m'} &= \langle \ell = 1, m|\, L_y \,|\ell = 1, m'\rangle \\ \left(L_z\right)_{m,m'} &= \langle \ell = 1, m|\, L_z \,|\ell = 1, m'\rangle \ . \end{aligned}$$

Again, write each of these in the usual form of a 3×3 matrix.

(c) (7) Call the last set of matrices L_i, with $i = 1, 2, 3$, and see if matrix multiplication gives – with your results for these matrices from part (b) –

$$[L_i, L_j] = i\hbar\, \varepsilon_{ijk}\, L_k$$

for each of the three i, j combinations.

4. In this problem $\vec{p}$ and $\vec{r}$ are Hilbert-space operators, $\vec{p}_0$ is a classical momentum, and $\vec{r}_0$ is a classical position vector.

In each part below please give a clear argument, justifying each step.

In the first two parts below, we wish to derive the operator identity

$$[\vec{r}, f(\vec{p})] = i\hbar\vec{\nabla}_{\vec{p}} f(\vec{p})$$

where $f(\vec{p})$ is an arbitrary function of the operator $\vec{p}$. You may assume the earlier results

$$\vec{p}\,|\vec{p}_0\rangle = \vec{p}_0\,|\vec{p}_0\rangle \quad \text{which implies that} \quad f(\vec{p})\,|\vec{p}_0\rangle = f(\vec{p}_0)\,|\vec{p}_0\rangle$$

$$\langle\vec{p}_0|\,\vec{r}\,|\psi'\rangle = +i\hbar\vec{\nabla}_{\vec{p}_0}\langle\vec{p}_0\,|\psi'\rangle \quad \text{where here it is useful to take} \quad |\psi'\rangle \equiv f(\vec{p})\,|\psi\rangle\ .$$

(a) (5) Show that

$$\langle p_0|\,[\vec{r}, f(\vec{p})]\,|\psi\rangle = i\hbar\vec{\nabla}_{\vec{p}_0}\left[f(\vec{p}_0)\,\langle\vec{p}_0|\psi\rangle\right] - f(\vec{p}_0)\left[i\hbar\vec{\nabla}_{\vec{p}_0}\langle\vec{p}_0|\psi\rangle\right]\ .$$

(b) (5) Then obtain the operator identity

$$[\vec{r}, f(\vec{p})] = i\hbar\vec{\nabla}_{\vec{p}} f(\vec{p})\ .$$

(c) (5) Using the result of part (b), show that

$$e^{i\vec{p}\cdot\vec{R}/\hbar}\,\vec{r}\,e^{-i\vec{p}\cdot\vec{R}/\hbar} = \vec{r} + \vec{R}\,.$$

where $\vec{R}$ is a numerical vector.

(d) (5) Show that

$$\langle\vec{r}_0|\,e^{-i\vec{p}\cdot\vec{R}/\hbar} = \langle\vec{r}_0 \pm \vec{R}|\ .$$

where you will determine the sign.

(e) (5) Show that the wavefunction

$$\phi(\vec{r}_0) \equiv \langle\vec{r}_0|\phi\rangle \quad , \quad |\phi\rangle \equiv e^{-i\vec{p}\cdot\vec{R}/\hbar}\,|\psi\rangle$$

is the same as the wavefunction

$$\psi(\vec{r}_0) \equiv \langle\vec{r}_0|\psi\rangle$$

except shifted by a distance $\vec{R}$. I.e., show that

$$\phi(\vec{r}_0) = \psi\left(\vec{r}_0 \pm \vec{R}\right)$$

where you will again determine the sign.

5. (5 points extra credit) Discuss the significance of the Laplace-Runge-Lenz vector in the history of astronomy and quantum mechanics, with specifics.